Belyi's theorem and dessins d'enfants

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Abstract

The group $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ is not understood as one would like and the Grothendieck's theory of "dessins d'enfants" will help us to understand it. This group acts on the set of abstract dessins faithfully and the dessins carry some combinatorial structure. If we find a total set of invariants for this action, it will help us to understand the group.
Let S be a compact and connected Riemann surface, and $f \in \mathcal{M}'(S)$ a locally non constant meromorphic function. It can be seen as a ramified cover of the sphere ramified over finitely many points and the complex structure in S is determined by f (section 1). It is possible to see $\mathcal{M}(S)$ as an extension of $\mathbf{C}$ of transcendental degree one thanks to f. We then recover the Riemann surface S by the compactification of $S'(P)$ the zero set of P irreducible in $\in \mathbf{C}[X, Y]$ (section 2). Belyi's theorem (4.2) said that S can be defined over $\overline{\mathbf{Q}}$ (ie we can find a polynomial $P \in \mathbf{Q}[X, Y]$ such that $S \simeq S(P)$) if and only if S is a ramified cover of the sphere Σ above at most three points. It is an important result that is at the beginning of the theory of dessins d'enfants (see [8]). The structure of the ramified cover is encoded in a bicolored graph, the so-called dessins d'enfants. We can find some algebraic equations to define the ramified cover (thanks to Belyi's theorem), and so the Galois group $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ acts on the set of dessins by acting on the coefficients defining our ramified cover. We prove Belyi's theorem in section 4 and discuss about the dessins and the action of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ in section 5.

Contents

Notations The usual fields of the rational numbers (respectively the real numbers, the complex numbers) are denoted by $\mathbf{Q}$ (respectively $\mathbf{R}$, $\mathbf{C}$). The sets of integers (respectively non negative integers) will be denoted by $\mathbf{Z}$ (respectively $\mathbf{N}$). We denote by Σ the Riemann sphere and by $\mathbb{D}$ the disc of radius 1 centered in 0 in $\mathbf{C}$. If S is a Riemann surface we denote by $\mathcal{M}(S)$ (respectively $\mathcal{M}'(S)$) the set of meromorphic functions on S (respectively meromorphic functions on S that are locally non-constants).

1 Riemann surfaces and ramified covers

1.1 Compact Riemann surfaces

Let S be a compact connected Riemann surface and let $p \in \mathcal{M}'(S)$. In this section we will discuss the regular and ramification points of p, and show that p is a finite cover of Σ minus a finite number of points.

An important setting is that there exists non constant meromorphic functions on S. One can see this by studying cohomology of sheaves, see [5].
The map p is proper since continue between two compact spaces. We will define the order of p at a zero $x_0 \in S$: take a chart (U, ϕ) in S centered at x_0, i.e. $\phi(x_0) = 0$, and define $V = p(U)$. The function p is holomorphic at x_0 means exactly that $p \circ \phi^{-1}$ is holomorphic from a neighbourhood of 0 in $\mathbf{C}$ to V. So we can write for $x = \phi^{-1}(w) \in U$

$$p(x) = \sum_{n=0}^{\infty} a_n w^n \ .$$

Since p is not constant, the a_n are not all zero, and we define the *order* of p at x_0 to be $n_0 = \inf\{n \in \mathbf{N} \mid a_n \neq 0\}$. We have in U

$$p(x) = w^{n_0} \sum_{n=0}^{\infty} a_n w^{n-n_0} = w^{n_0} f(w) \ ,$$

with f holomorphic and non constant in a neighbourhood of 0, and $f(0) = a_{n_0} \neq 0$. So there exist a neighbourhood U' of 0 in $\mathbf{C}$ and a holomorphic function g in U' such that $g^{n_0} = f$. Then, let $z = \psi(w) = wg(w)$ define in U'. We have $\psi'(0) = g(0)$ is an n_0^{th} root of a_{n_0} so it is not zero and our function ψ is a biholomorphism in a neighbourhood of 0 in $\mathbf{C}$, and so $\psi \circ \phi$ is a biholomorphism from a neighbourhood U'' of x_0 in S to a neighbourhood of 0 in $\mathbf{C}$, that is a chart of S centered in x_0, with

$$p(x) = z^{n_0} \ ,$$

for all $x = (\psi \circ \phi)^{-1}(z) \in U''$. Then we see that the number n_0 is unique : it is determined by the number of preimages in U'' of a complex number close enough to 0.

We define, for $x \in S$, the *order* n_x of p at x to be the order of $p - p(x)$ at x (if $p(x) \neq \infty$) or the order of $1/p$ at x (if $p(x) = \infty$). If $n_x \geq 2$ we say that x is a *ramification point* or a *branch point* of p, that p *ramifies* at x, and we say that $y = p(x) \in \Sigma$ is a *ramification* (or *branch*) *value* of p. Otherwise, if $n_x = 1$, x is said to be a *regular point*. We denote by $\Delta(p) \subset \Sigma$ the set of ramification values of p. Let us define the *order* n of p to be the sum of the orders of p at its zeros.

At a ramification point $x \in S$, by writing in a neighbourhood of x

$$p(y) - p(x) = z^{n_x} \ ,$$

where $z = \phi(y)$ is the coordinate of an adapted chart ϕ, we see that p ramifies at x if and only if its derivative p' vanishes at x. Since p is meromorphic and not constant, p' is meromorphic and not zero, so its zeros are isolated. Thus the zeros of p' are in finite number since S is compact and we just proved the following :

Proposition 1.1. *Let $f : X \to Y$ be a proper non constant meromorphic function between two Riemann surfaces. The set Δ of ramification values of f is closed and discrete in Y. Moreover, if Y is compact (and so X must be compact too), Δ is finite.*

Proposition 1.2. *If $g : S \to S'$ and $f : S' \to S''$ are meromorphic, non-constant, then the ramification values of $f \circ g$ are included in the union of two sets :*
- the set of the images by f of the ramification values of g,
- the set of the ramification values of f.

Proof. One can see this through the chain rule since at a ramification point the derivative vanishes. □

Definition 1.3. *Let $\Sigma' = \Sigma \setminus \{y_1, \dots, y_k\}$ be the set of regular values of p.*

Proposition 1.4. *The map*

$$p_{|p^{-1}(\Sigma')} : p^{-1}(\Sigma') \to \Sigma' ,$$

is a finite cover of order n (the order of p).

Proof. If $y \in \Sigma'$, then for all $x \in p^{-1}(y)$, p is regular at x and so is a local homeomorphism (in fact a local biholomorphism). The map p is proper, surjective, and is a local homeomorphism, so it is a finite cover (of a certain order $m \in \mathbf{N}^*$) of Σ'.
Since $p_{|\Sigma'}$ is a cover, each value of $p_{|\Sigma'}$ is taken m times. In a branch value $y \in \Sigma$, let $p^{-1}(y) = \{x_1, \dots, x_k\}$, and let $U_1, \dots, U_k$ be disjoints open neighbourhoods of $x_1, \dots, x_k$ such that

$$p(x') = z_i^{n_{x_i}} + y ,$$

with $z_i = \phi_i(x')$ is a chart in U_i. Shrinking the U_i's we can assume that $p(U_i) = V$ for all i with V neighbourhood of y not containing some other branch value. Then wee see that every value $y' \in V \setminus \{y\}$ is regular so taken m times. In the other hand y is taken k times with multiplicities $n_1, \dots, n_k$, and the number of preimages of y' in each U_i is exactly n_i, so

$$m = n_1 + \ldots + n_k = n$$

where n is the order of p. □

Remark 1.5. *Note that we proved the following fact :*
Each value of p is taken n times (counted with multiplicities).

Proposition 1.6. *Let X, Y be two compact Riemann surfaces, and $\Delta_X \subset X$, $\Delta_Y \subset Y$ finite sets. If $f : X \setminus \Delta_X \to Y \setminus \Delta_Y$ is an isomorphism, then we can extend it on an isomorphism $\tilde{f} : X \to Y$.*

Proof. If $x \in \Delta_X$, take a disc U around x that does not contain any other point of Δ_X. If $(x_n)_{n \in \mathbf{N}}$ is a sequence of U that converges to x, then $(y_n)_n = (f(x_n))_n$ is a sequence of $Y \setminus \Delta_Y$ that converges to a $y \in \Delta_Y$ (otherwise $f^{-1}(y) = x$, impossible since f is an isomorphism). Up to shrink U, $f(U)$ contains a unique $y \in \Delta_Y$. By looking at each point of Δ_X, we see that Δ_X is in one-to-one correspondence with Δ_Y.
So f is bounded near x and we can (by Riemann removable singularity theorem) extend it in a unique way by making $\tilde{f}(x) = y$. Doing that at every point $x \in \Delta_X$ we obtain a holomophic function $\tilde{f} : X \to Y$ of degree one, i.e. an isomorphism. □

We end this section with a famous theorem that will be useful. Let us recall that all Riemann surface is homeomorphic to a "torus with g holes" where g is the "genus" of S and it is a topological invariant. Another important number is the Euler-Poincaré characteristic $\chi = 2 - 2g$ and the fact that for every triangulation of our surface, if we count the number of vertices v, the number of edges e and the number of faces f of the triangulation, then $\chi = v - e + f$.

Theorem 1.7 (Riemann-Hurwitz formula)**.** *Given a holomorphic map $p : S \to S'$ of order n between two compact Riemann surfaces, and Euler-Poincaré characteristic χ and χ' then we have*

$$\chi = n\chi' - \sum_{x \in p^{-1}(\Delta)} (n_x - 1) ,$$

where $\Delta \subset S'$ is the set of ramification values and n_x the order of ramification of the point x.

It can be proved in many different manners and we just give here a sketch from a topological point of view.

Sketch of a proof. Let us take a triangulation of S' (with v' vertices, e' edges and f' faces) such that the ramification values Δ are some vertices of this triangulation. We can then pull-back this triangulation to S. If p does not ramify, then it is a local biholomorphism and so the triangulation in S will have $v = nv'$ vertices, $e = ne'$ edges and $f = nf'$ faces. If p ramifies at a point x (that is a vertex of the triangulation in S), then looking in a good chart around x and $y = p(x)$, p is written $z \mapsto z^{n_x}$ so in the pullback triangulation there is only one vertex in x, and $n_x * e_y$ edges starting from x where e_y is the number of edges starting from y, and $n_x * f_y$ faces that touch x. But since the sum of the orders of ramification of all the points above y is equal to n, when computing χ' from the image triangulation we find that

$$\begin{aligned} v &= nv' - \sum_{x \in p^{-1}(\Delta)} (n_x - 1) , \\ e &= ne' , \\ f &= nf' , \end{aligned}$$

and so

$$\chi = n\chi' - \sum_{x \in p^{-1}(\Delta)} (n_x - 1) .$$

□

1.2 Ramified covers

In general, we can define what is a ramified cover between topological surfaces. We will see in this section that finite covers are equivalent to ramified covers up to add some points.

Definition 1.8. *Let X and Y be two topological surfaces, and $p : X \to Y$ be a continuous mapping. The map p is said to be a* ramified cover *if the following holds :*

- *The map p is proper*

- *For every $x \in X$, there exist $d_x \in \mathbf{N}^*$, a chart (U, z) centered in x and a chart (V, w) centered in $y = p(x)$ such that in these charts we have*

$$w = p(z) = z^{d_x} \ .$$

If $x \in X$, we will call d_x the order of x. If $d_x > 1$ then x is called a ramification *(or a* branch*)* point *and $y = p(x)$ a* ramification *(or a* branch*)* value*. Otherwise x is called a* regular point.
We will call the opens U and V normal *neighbourhoods of x and y respectively.*
Given two ramified covers $p : X \to Y$ and $p' : X' \to Y$ we say that $f : X \to X'$ is a morphism of ramified covers *if it is continuous and fiber preserving (i.e. $p' \circ f = p$).*

The integer d in this definition does not depend on the choice of the chart since for all $z \in p(U) \setminus p(x)$, $p^{-1}(z)$ consists of exactly d points.

Proposition 1.9. *Let $p : X \to Y$ be a ramified covering. The set $\Delta(p)$ of ramification values of p is closed and discrete in Y. Moreover, if Y is compact (and so X must be compact too), $\Delta(p)$ is finite.*

Proof. Let $x \in X$, $y \in Y$. If we take a normal neighbourhood U of x, we see that in $U \setminus \{x\}$ no point can be a ramification point, so the set of ramification points is discrete. The set $p^{-1}(\{y\})$ is discrete by the same kind of argument, so it is finite since p is proper. Let us denote $\{x_1, \dots, x_m\} = p^{-1}(\{y\})$, and $U_1, \dots, U_m$ disjoint normal neighbourhoods of $x_1, \dots, x_m$ respectively such that p is written is these maps $z \mapsto z^{d_i}$ from U_i to $V_i = p(U_i)$. Taking $V = \cap V_i$ and up to shrink the U_i we can suppose that they are disjoints and that $p(U_i) = V$. Since $p_{|U_i}$ are written like that, there is no ramification point in the neighbourhood of the x_i's so $V \setminus \{y\}$ doesn't contains any branch value and so $\Delta(p)$ is discrete, and closed because locally finite. □

If $p : X \to Y$ is a ramified cover, and if X connected, the degree of p is the number of preimages of a regular value $y \in Y \setminus \Delta(p)$. This doesn't depend of the choice of y (this will be a consequence of theorem 1.11).

Let us recall an important setting that comes from the classification theorem of covers :

Proposition 1.10. *Every connected and finite cover of $\mathbb{D}^* = \mathbb{D} \setminus \{0\}$ is isomorphic to*

$$\begin{aligned} \mathbb{D}^* &\to \mathbb{D}^* \ , \\ z &\mapsto z^d \ . \end{aligned}$$

Proof. The fundamental group $\pi_1(\mathbb{D}^*)$ is isomorphic to $\mathbb{Z}$ since the circle $\mathbb{S}^1$ is a retract by deformation of $\mathbb{D}^*$. We can say also that the universal covering of this space is the left half plane $\{z \in \mathbf{C} \mid \Re(z) < 0\}$ with the exponential map exp. So the connected covers of $\mathbb{D}^*$ are in bijection with the subgroups of $\mathbf{Z}$ (up to conjugation but $\mathbf{Z}$ is abelian) which are co-finite. The subgroups of $\mathbf{Z}$ are $\{0\}$ and $d\mathbf{Z}$ for a $d \in \mathbf{N}^*$. But we know that

$$\begin{aligned} \mathbb{D}^* &\to \mathbb{D}^* \ , \\ z &\mapsto z^d \ , \end{aligned}$$

is a cover with covering group $\mathbf{Z}/d\mathbf{Z}$. □

This allows us to claim the most important theorem of this section :

Theorem 1.11. *Let Y be a topological surface, and Δ be a discrete and closed subset of Y. There is a one-to-one correspondence between finite ramified covers $p : X \to Y$ ramified above a subset of Δ and finite covers $p : X' \to Y'$ where $Y' = Y \setminus \Delta$. This correspondence is functorial.*

Proof. If $p : X \to Y$ is a ramified cover of degree n ramified over a subset of Δ, we note $X' = X \setminus p^{-1}(\Delta)$. Let's take $y \in Y'$ and $\{x_1, \ldots, x_n\} = p^{-1}(\{y\})$. Since the x_i's are not ramified points, and by the definition, they are disjoints neighbourhoods U_i of each x_i and V of y such that $p_{|U_i}$ is a homeomorphism from U_i to V. Thus $p^{-1}(V) = \sqcup U_i$ and $p' = p_{|X'}$ is a cover of Y'. Moreover if $p : X \to Y$ and $\hat{p} : \hat{X} \to Y$ are finite ramified covers and $f : X \to \hat{X}$ is a fiber preserving mapping then the restriction of f to X' will be fiber preserving and valued in $\hat{X}'$. So $f_{|X'}$ will be a morphism of covers.

Now if $p' : X' \to Y'$ is a finite cover, with X' topological surface, then we will extend it to a ramified cover of Y. Let's take $y \in \Delta$, and a chart (V, ψ) in the neighbourhood of y such that $\psi(V) = \mathbb{D} \subset \mathbf{C}$, $\psi(y) = 0$ and $(V \setminus \{y\}) \cap \Delta = \emptyset$. We define $V' = V \setminus \{y\}$. Then $p'_{|V'}$ is a cover so $\psi \circ p'_{|\mathbb{D}^*}$ too. Let's take U' a connected component of $p'^{-1}(V')$ (there is a finite number of connected component above V' because p' is of finite degree). By proposition 1.10 it must be a homeomorphism $\phi : U' \to \mathbb{D}^*$ with

$$\begin{aligned} \psi \circ p' \circ \phi^{-1} : \mathbb{D}^* &\to \mathbb{D}^* , \\ z &\mapsto z^{d_1} . \end{aligned}$$

Then we add a point $x = x_{y,1}$ to X' such that we continue ψ as a chart with $\psi(x) = 0$. We need to check that this chart glue with the charts of X', and for that we only need to check the gluing in $U = U' \cup \{x\}$. The gluing holds since ψ is a homeomorphism in U'. We can add a point in each connected component of $p^{-1}(V)$, and doing this for all point of Δ we get a new surface $X = X' \cup \{x_{y,i} \mid y \in \Delta,\ 1 \le i \le n(y) \le n\}$ and we can extend p' through X by defining $p(x_{y,i}) = y$.
Now $p : X \to Y$ is a ramified covering :
In fact since $p_{|X'}$ is a covering, we need just to check the definition of ramified coverings in the $x_{y,i}$'s (obvious by definition of the $x_{y,i}$'s) and to show that p is proper. First, p is open since it is in X' and since $z \mapsto z^d$ is open for all $d \in \mathbf{N}^*$. Let's take $K \subset Y$ a compact set and let us prove that $p^{-1}(K)$ is compact. Let $(U_i)_{i \in I}$ be an open cover of $p^{-1}(K)$. K is a compact of Y so $K \cap \Delta = \{y_1, \ldots, y_k\}$ is finite. Let us take $(V_1, \psi_1), \ldots, (V_k, \psi_k)$ charts centered at the y_i's like in the definition of p (the $\psi_i : V_i \to \mathbb{D}$ are homeomorphisms) and denote by K_i the set $\psi_i^{-1}\left(\overline{\mathbb{D}(0, 1/2)}\right) \cap K$. $K_1, \ldots, K_k$ are compact or empty, and p is written above the V_i's

$$\begin{aligned} \psi_i \circ p' \circ \phi^{-1} : \mathbb{D}^* &\to \mathbb{D}^* , \\ z &\mapsto z^{d_1} , \end{aligned}$$

in neighbourhood of each $x_{y_i,j}$, so $p^{-1}(\cup K_i) = \sqcup p^{-1}(K_i)$ is compact in X (or empty). Moreover, let's note $K' = K \setminus \cup \psi_i^{-1}(\mathbb{D}(0, 1/2))$. Since $p' = p_{|X'}$ is proper (finite covering) and $K' \subset X'$, $p'^{-1}(K')$ is compact. But $K = K' \cup K_1 \cup \ldots \cup K_k$ so $p^{-1}(K) = p^{-1}(K') \cup p^{-1}(\cup K_i)$ is compact.
If $p' : X' \to Y'$ and $\hat{p} : \hat{X}' \to Y'$ are two finite covers, and $f' : X' \to \hat{X}'$ is a cover morphism, then denoting by X and $\hat{X}$ the completions of X' and $\hat{X}'$ respectively as

above, we want to define f from X to $\hat{X}$ that continue f'.
Let $x \in X$ be such that $p(x) = y \in \Delta$, and let us denote by $\hat{x}_1, \dots, \hat{x}_k$ the preimages of y by $\hat{p}$. Let $\left(\hat{U}_i\right)_{1 \le i \le k}$ be disjoint open neighbourhoods of the $\hat{x}_i$'s, and let V be an open neighbourhood of y in Y such that $\hat{p}^{-1}(V) \subset \cup \hat{U}_i$. Let $U \subset X$ be a normal neighbourhood of x, and such that $p(U) \subset V$. We have that $U \setminus \{x\}$ is connected because homeomorphic to $\mathbb{D}^*$ and it is in $p^{-1}(Y \setminus \Delta)$, so $f(U \setminus \{x\})$ is a well define connected open included in $\cup \hat{U}_i$. Thus $f(U \setminus \{x\})$ must belongs to only one $\hat{U}_i$, say $\hat{U}_1$, and so we define $f(x) = \hat{x}_1$. This is continuous in $\hat{x}_1$ and so by doing this in each point of $p^{-1}(\Delta)$ we define a continuous function

$$f : X \to \hat{X}\,,$$
$$x \mapsto \begin{cases} f(x) = \hat{x} \in \hat{p}^{-1}(\Delta) & \text{like above, if } x \in p^{-1}(\Delta) \\ f'(x) & \text{if } x \notin p^{-1}(\Delta) \end{cases}\,,$$

which is trivially fiber preserving. The proof shows us that this construction must be unique, and by definition $f_{|X'} = f'$. ◻

With this theorem we want to make the following definition.

Definition 1.12. *The ramified cover $f : X \to Y$ is said to be* Galois *if the associated cover $f' : X' \to Y'$ is Galois.*

After this equivalence between ramified covers and covers of the basis minus some points, we will see an equivalence with analytic ramified covers.

Lemma 1.13. *Given a commutative diagram*

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[d, "\alpha"'] & S' \arrow[d, "\beta"] \\ B \arrow[r, "g"'] & B' \end{tikzcd}$$

with S, S', B, B' connected Riemann surfaces, f continuous, α, β, g holomorphic and β non constant. Then f is holomorphic.

Proof. Let $p \in S$ and prove that f is analytic in p. We have that in some charts around $q = f(p)$ in S' and $\beta(q)$ in B', β is written

$$V \to W\,,$$
$$z \mapsto z^d\,,$$

with $d \in \mathbf{N}^*$. Since f is continuous, $f^{-1}(V)$ is an open U of S containing p. In this open, we have

$$f^d = \beta \circ f = g \circ \alpha\,.$$

The map f^d is holomorphic from U to V so its zeros $Z = \{x \in U \mid f(x) = 0\}$ are isolated or f is constant in U. Then since β is invertible at each point of $V \setminus \{0\}$, f is (constant or) holomorphic in $U \setminus Z$ with Z closed and discrete, so is holomorphic in U. ◻

Theorem 1.14. *If $p : X \to S$ is a ramified cover between a topological surface X and a Riemann surface S, then there is a unique complex structure on X such that p is homolorphic.*

Proof. Take an atlas $(U_{i,j}, \phi_{i,j})_{i\in I,\ j\in J_i}$ of X and an atlas $(V_i)_{i\in I}$ of B of normal neighbourhoods such that p is written in each pair $(U_{i,j}, V_i)$ with $i \in I$ and $j \in J_i$

$$z \mapsto z^{d_{i,j}} .$$

Then let us check that the transition maps are holomorphic. Observe the intersection of two different opens $U_{i,j}$ and $U_{i',j'}$ does not contain ramification points. Let's denote $V = p(U)$. Let $x \in U$, $y = p(x) \in V$ and D be a little "disc" around y contained in V. We have that $p^{-1}(D)$ is a disjoint union of "discs" in U so only one, let say B, contains x. We have now this picture

$$\begin{tikzcd} B \arrow[r, "\varphi"] \arrow[d, "\mathrm{id}"'] & B \arrow[d, "\mathrm{id}"] \\ D \arrow[r, "\psi"'] & D \end{tikzcd}$$

with φ the transition map in X that is continuous, ψ the transition map in S that is holomorphic. Then by the previous lemma, φ is holomorphic and the atlas is a holomorphic one, that is $(U_{i,j}, \phi_{i,j})_{i,j}$ is a complex structure on X.
If we take another atlas of X making p holomorphic, then the map

$$\mathrm{id} : \mathrm{X} \to \mathrm{X}$$

is a biholomorphic map since it may be verified locally, using normal neighbourhoods. □

Definition 1.15. *The* monodromy *of a finite cover $p : X \to B$ is the representation of the fundamental group of the basis $\pi_1(B, b)$ into the permutation group $\mathfrak{S}_{X(b)}$ of the fiber $X(b) = p^{-1}(b)$. The action of $\pi_1(B, b)$ on $X(b)$ comes from path lifting from B to X.*

We can see the monodromy group as $\pi_1(B, b)/p_*\pi_1(X, x)$ with $x \in p^{-1}(b)$ and this is well-defined up to choose another point of the fiber (which give us a conjuged of $p_*\pi_1(X, x)$). The monodromy determine uniquely the cover, up to isomorphisms, if the basis B is connected. So since a ramified cover (S, p) of the sphere ramified over $\{0, 1, \infty\}$ can be seen as a cover of $\Sigma \setminus \{0, 1, \infty\}$, then ask what could be the monodromy of the cover, it will determine uniquely up to isomorphisms (S, p). But be careful : only the monodromy group doesn't determine the cover. For this (and much more) one can look at [3].

Proposition 1.16. *The fundamental group Γ of Σ minus the three points $0, 1, \infty$ is*

$$\pi_1(\Sigma \setminus \{0, 1, \infty\}) = \langle \rho_0, \rho_1, \rho_\infty \mid \rho_0\rho_1\rho_\infty = e\rangle ,$$

where ρ_x is the homotopy class of a loop originating from 1/2 and going clockwise around x.
The group Γ is isomorphic to $\mathbf{Z}^*\mathbf{Z}$ where $\mathbf{Z}^*\mathbf{Z}$ is the free group generated by two elements.

So the monodromy group of p finite cover of Σ will be isomorphic to a quotient of $\mathbf{Z}^*\mathbf{Z}$ by a subgroup of finite index.

2 Algebraic functions

2.1 Riemann surfaces associated to algebraic functions

Riemann surfaces appeared, thanks to Bernhard Riemann, in order to give meaning to functions such as $z \mapsto z^{1/d}$ which can't be defined holomorphically (even continuously) in $\mathbf{C}$ or $\mathbf{C}^*$. He made for this a construction of a surface where it is possible to continue functions along paths. It is the origin of Riemann surfaces, and this will lead to the theory of covers. In this section we make this construction, and prove that the projection on the first coordinate is a finite cover.
This section is partially inspired of [7] part X, and of [5].

Definition 2.1. *An* algebraic function *is an irreducible polynomial* $P \in \mathbf{C}[X, Y]$.

It can be seen as a "function" since for every $z \in \mathbf{C}$ the polynomial $P(z, Y)$ admits some roots $\zeta(z)$ "depending" on z. For example the polynomial $P(X, Y) = X - Y^3$ represent the function $\zeta(z) = z^{1/3}$ since all (z, ζ) must verify $P(z, \zeta) = 0$ so $z = \zeta^3$. We can point the fact that in this example the problem is that we want to write ζ as a function of z, because in the other side it works good : $z = \zeta^3$ is holomorphic in ζ. But in the general case both sides are problematic. Our construction will be able to work in the too sides and for all algebraic function.

Let us take an algebraic function of degree n in Y

$$P(X, Y) = P_0(X)Y^n + \ldots + P_{n-1}(X)Y + P_n(X) .$$

Proposition 2.2. *The set* $\Delta(P) = \{z \in \mathbf{C} \mid P(z, Y)$ *has less than n distincts roots*$\}$ *is finite.*

Proof. If $z \in \mathbf{C}$, the polynomial $Q(Y) = P(z, Y)$ is of degree n except if z is a root of $P_0(X)$, otherwise it has multiple roots only if it is not coprime to its derivative $Q'(Y)$. The tool we need to use is the discriminant[1] $\delta(Q) = \delta_Y(P(z, Y))$ of Q, a polynomial in its coefficients (which depend on z) which will vanish if and only if it has multiple roots. So $z \in \Delta(P)$ if and only if z is a root of $P_0(X)$ or is a root of $\delta_Y(P(X, Y))$, so there is only a finite number of such z. □

Since $\Delta(P)$ is finite, it is discrete in $\mathbf{C}$.

Definition 2.3. *The Riemann surface associated to P is*

$$S'(P) = \{(z, \zeta) \in \mathbf{C}^2 \mid P(z, \zeta) = 0 \textit{ and } z \in \mathbf{C} \setminus \Delta(P)\} .$$

Proposition 2.4. *The map*

$$\mathbf{z} : S'(P) \to \mathbf{C} \setminus \Delta(P) ,$$
$$(z, \zeta) \mapsto z ,$$

is a cover.

Proof. First, $S'(P)$ is a Riemann surface, and $\mathbf{z}$ is locally biholomorphic. One can see this through implicit function theorem, since P is a holomorphic map from $\mathbf{C}^2$ to $\mathbf{C}$ with $d_2P(z, \zeta) \neq 0$ for all $(z, \zeta) \in S'(P)$. If $z_0 \in \mathbf{C} \setminus \Delta$, the set $\{\zeta \in \mathbf{C} \mid P(z_0, \zeta) = 0\}$ contains n elements $\{\zeta_1, \ldots, \zeta_n\}$, so there exists disjoints neighbourhoods U_i of ζ_i in $\mathbf{C}$ and V of

[1] It is the resultant between Q and Q'

z_0 in $\mathbf{C} \setminus \Delta$, and holomorphic functions $f_i : V \to U_i$ such that for all $(z, \zeta) \in V \times U_i$, $(z, \zeta) \in S'(P)$ if and only if $\zeta = f_i(z)$. So over the neighbourhood V of z_0 we can write

$$\mathbf{z}^{-1}(V) = \bigsqcup_i U_i = \bigsqcup_i f_i(V) ,$$

and since $\mathbf{z}_{|U_i} = (\mathrm{id} \times \mathrm{f_i})^{-1}$, $\mathbf{z}$ is locally biholomorphic. We so have that $\mathbf{z}$ is an analytic cover of $\mathbf{C} \setminus \Delta(P) = \Sigma \setminus (\Delta(P) \cup \{\infty\})$. □

By the same way that in section 1.2, we can transform this cover on a ramified cover of a compact Riemann surface $S(P)$ over Σ and in a unique way, by adding some points to $S'(P)$. We will continue to note $\mathbf{z} : S(P) \to \Sigma$ the ramified cover.

Proposition 2.5. *The surface $S(P)$ is connected.*

Proof. We only need to prove that $S' = S'(P)$ is connected. We can define the elementary symmetric functions $\sigma_1, \ldots, \sigma_n \in \mathcal{M}(S(P))$ of $f = \mathbf{z}$ in $\mathbf{C} \setminus \Delta(P)$, see lemma 2.10. In fact we can separate them by taking the elementary symmetric functions $\{\sigma_{k,1}, \ldots, \sigma_{k,n_k}\}$ of $f^{(k)} = f_{|X_k}$ in each connected component X_k of S' of degree n_k over $\mathbf{C} \setminus \Delta(P)$. They must locally satisfy two equations :

$$\prod_{1 \le i \le n_k} (Y - f_i^{(k)}) = Y^n - \sigma_{k,1} Y^{n-1} + \ldots + (-1)^n \sigma_{k,n_k} ,$$

for each k, and

$$\prod_{1 \le i \le n} (Y - f_i) = Y^n - \sigma_1 Y^{n-1} + \ldots + (-1)^n \sigma_n ,$$

with f_i and $f_i^{(k)}$ the local sections of f, i.e. $f_i(z)$ is some root ζ_i of $P(z, Y)$.
But we have

$$\prod_k \prod_{1 \le i \le n_k} (Y - f_i^{(k)}) = \prod_{1 \le i \le n} (Y - f_i) = Q(z, Y) ,$$

with $Q = P/P_0 \in \mathbf{C}(X)[Y]$, and this globally in $\mathbf{C}(z)[Y]$. But P is irreducible in $\mathbf{C}[X][Y]$ so it is in $\mathbf{C}(X)[Y]$ by Gauss' lemma, and so Q must be irreducible too. The last equation implies $k = 1$, i.e. $S'(P)$ is connected. □

By the same idea than $\mathbf{z}$, ζ can be seen as a meromorphic function on $S(P)$. Indeed we construct another $S''(P)$ by the same way but with ζ. Since $\Delta(P)$ and $\Delta''(P) = \{\zeta \in \mathbf{C} \mid P(X, \zeta)$ has less than n distincts roots$\}$ are finite, the two compactified Riemann surfaces that arise from z or from ζ are isomorphic : Let us denote $\Delta = \Delta(P) \cup \Delta''(P)$. The equality

$$\begin{aligned} S'(P)_{|\mathbf{C} \setminus \Delta} &\to S''(P)_{|\mathbf{C} \setminus \Delta} \\ (z, \zeta) &\mapsto (z, \zeta) , \end{aligned}$$

is an isomorphism, so by proposition 1.6 their compactification are isomorphic too. We will continue to call them $S(P)$.

So $S(P)$ is endowed with two ramified covers that come from the projections on the two coordinates.

$$\begin{tikzcd} S'(P) \arrow[r, "\mathrm{in}"] \arrow[d, "\pi_1"'] & S(P) \arrow[dl, "z"] \arrow[dr, "\zeta"'] & S''(P) \arrow[l, "\mathrm{in}"'] \arrow[d, "\pi_2"] \\ \Sigma & & \Sigma \end{tikzcd}$$

We will see in the next section that every meromorphic function on $S(P)$ is a rational function of these two "coordinates" meromorphic functions.

2.2 The field of meromorphic functions

In this paragraph, S will be a compact connected Riemann surface. We will see here that $\mathcal{M}(S)$ is a function field, and recover some algebraic function defining our Riemann surface.
First let us prove :

Proposition 2.6. *The field $\mathcal{M}(\Sigma)$ of meromorphic functions on the Riemann sphere is isomorphic to $\mathbf{C}(z)$.*

Proof. Let $g \in \mathcal{M}(\Sigma)$. It has a finite number of poles $\{z_1, \ldots, z_k\}$, and for each pole z_i, let us denote the polar par of g at z_i by $P_i(\frac{1}{z-z_i})$ if z_i is finite or $P_i(z)$ if $z_i = \infty$, with $P_i \in \mathbf{C}[X]$ for all $i \in \{1, \ldots, k\}$. Then the function $h = g - P_1 - \ldots - P_k$ is holomorphic in Σ so is constant and g is equal to a rational function. □

The most important theorem here is his one :

Theorem 2.7. *Let $f \in \mathcal{M}'(S)$ and denote by $n \in \mathbf{N}^*$ its order. The map*

$$\begin{aligned} f^* : \mathbf{C}(z) = \mathcal{M}(\Sigma) &\longrightarrow \mathcal{M}(S) \\ \alpha &\longmapsto f^*\alpha = \alpha(f)\,, \end{aligned}$$

is a field homomorphism. The field $\mathcal{M}(S)$ is an extension of $\mathbf{C}(z)$ of degree $[\mathcal{M}(S) : \mathbf{C}(z)] = n$.

To prove this theorem we will need few lemmas :
First, given $f \in \mathcal{M}'(S)$ and $g \in \mathcal{M}(S)$ we will define the *elementary symmetric functions* of g with respect to f. Since f is a cover outside a finite set of points $\Delta(f)$, we can look at what happens near a regular value $y \in \Sigma$. Let's set $\{x_1, \ldots, x_n\} = f^{-1}(y)$ and take some neighbourhoods U_i of each x_i and V of y such that for $1 \leq i \leq n$, $f_{|U_i}$ are biholomorphisms between U_i and V. Then, let's define (for each $1 \leq i \leq n$) $\tau_i = {f_{|U_i}}^{-1}$ and $g_i = g \circ \tau_i \in \mathcal{M}(V)$.

Definition 2.8. *The* elementary symmetric functions *of g (with respect to f) in V are*

$$\sigma_k(g_1, \ldots, g_n) := \sum_{1 \leq i_1 < \ldots < i_k \leq n} \left(\prod_{1 \leq l \leq k} g_{i_l} \right),$$

for $1 \leq k \leq n$.

A computation shows the following

Proposition 2.9. *The elementary symmetric functions of g (with respect to f) in V satisfy*

$$\prod_{1 \leq i \leq n} (X - g_i) = X^n - \sigma_1 X^{n-1} + \ldots + (-1)^n \sigma_n\,. \tag{1}$$

Lemma 2.10. *The elementary symmetric functions $(\sigma_k)_{1 \leq i \leq n}$ can be defined globally as meromorphic functions on Σ.*

Proof. We can define the σ_k's in neighbourhoods of all $y \in \Sigma \setminus \Delta(f)$. These functions glue to meromorphic functions $\sigma_k \in \mathcal{M}(\Sigma \setminus \Delta(f))$. Since g is well defined in all S, then the σ_k can be meromorphically continued through Σ :
Let take a ramification value $y \in \Delta(\Sigma)$, and $\{x_1, \ldots, x_m\} = f^{-1}(y)$. They are charts (ϕ_i, U_i) of S centered in each x_i and (ψ, V) centered in y with $f^{-1}(V) = \cup U_i = U$ such that in U_i the map f become

$$z \mapsto z^{n_i} ,$$

where n_i is the order of f at the point x_i. Since ψ vanishes at y, the function $(f^*\psi)^k g = \psi(f)^k g$ defined in U is holomorphic in each x_i as soon as k is large enough. But look at the symmetric functions of $\psi(f)^k g$ in V

$$\prod_{1 \le k \le n} (X - \psi^k g_i) = X^n - \psi^k \sum_i g_i + \ldots + (-1)^n (\psi^k)^n \prod_i g_i$$
$$= X_n + \psi^k \sigma_1 + \ldots + \psi^{kn} \sigma_n .$$

We have $\sigma_i(\psi(f)^k g) = \psi^{ki}\sigma_i(g)$ and since $\psi(f)^k g$ is bounded near each x_i, the $\sigma_i(\psi(f)^k g) = \psi^{ki}\sigma_i(g)$ are bounded near y and we can continue them holomorphically through y, and these means that the σ_i can be continued meromorphically through y, so through all Σ, and that ends the proof of the lemma. □

Lemma 2.11. *Given a meromorphic function $g \in \mathcal{M}(S)$, and $(\sigma_i)_{1 \le i \le n}$ its elementary symmetric functions, then*

$$g^n + (f^*\sigma_1) g^{n-1} + \ldots + f^*\sigma_n = 0 ,$$

that is, g is of degree less than n in $\mathcal{M}(S)$.

Proof. The equation of g follows from the definition of the elementary symmetric functions because f is locally biholomorphic near each regular point x, so the equation given here is just the pull-back of (1) by f, and evaluation in g. Since it is true on the regular points, it is true in S by analytic continuation. □

Lemma 2.12. *Given a field extension $\mathbf{k} \subset \mathbf{K}$ between fields of characteristic 0, if every element in $\mathbf{K}$ is of degree less than $n \in \mathbf{N}^*$, then the extension is of degree less than n.*

Proof. Let $x \in \mathbf{K}$ be an element of maximal degree $m \le n$ over $\mathbf{k}$, then if $x' \in \mathbf{K}$ let us look at $\mathbf{k}[x, x']$ which is included in $\mathbf{K}$:
By the primitive element theorem, there exist $y \in \mathbf{K}$ such that $\mathbf{k}[x, x'] = \mathbf{k}[y]$ but y must be of degree less than m by definition, and $\mathbf{k}[x, x']$ must be an extension of degree more than m, so these degrees are equal to m and since $\mathbf{k}[x, x'] = \mathbf{k}[x][x']$ and $\mathbf{k}[x]$ is already of degree m, x' is of degree 1 over $\mathbf{k}[x]$, i.e. $x' \in \mathbf{k}[x]$ and $\mathbf{k}[x] = \mathbf{K}$. □

Lemma 2.13. *Let $x_1, \ldots, x_n$ be n points of S, then there exists a meromorphic function $g \in \mathcal{M}(S)$ that separate these points, i.e. $g(x_i) \neq g(x_j)$ for all $i \neq j$.*

We will accept this statement since it depends of theory that we don't want to discuss here (again see [5] corollary 14.13, and all paragraph 14).
So we are now able to prove the theorem :

Proof of the theorem. The map f^* is a field homomorphism.
The three first lemmas show us that $\mathcal{M}(S)$ is an extension of degree less than n of $\mathbf{C}(z)$.
Let us take $y \in \Sigma \setminus \Delta(f)$ and, thanks to lemma 2.13, a function $g \in \mathcal{M}(S)$ which separate points of $f^{-1}(y) = \{x_1, \ldots, x_n\}$. Then the degree of g over $\mathbf{C}(z)$ is n :
An equation of g will vanish at the n distinct preimages of y so the degree of this equation must be at least n, and so $\deg(g)$ is n. □

The field $\mathcal{M}(S)$ characterizes S : Thanks to f we can write $\mathcal{M}(S) = \mathbf{C}(f)[g]$ (before we wrote $\mathbf{C}(z)$ but z was seen as a function of $\mathcal{M}(S)$ through f^*) where g is a primitive element of the extension. Let's take a function g that separate the n preimages of a regular value y like in the proof of the theorem 2.7, it will be useful. Let $Q \in \mathbf{C}(X)[Y]$ be the minimal polynomial of g. There is $P_0 \in \mathbf{C}[X]$ so that $P = P_0 Q$ is an irreducible polynomial in $\mathbf{C}[X][Y]$ of degree n in Y.
More precisely, let

$$Q(X,Y) = \sum_{0\le i\le n} Q_{n-i}(X)Y^i \ ,$$

where $Q_i = a_i/b_i \in \mathbf{C}(X)$ satisfy for all i :

- $pgcd(a_i, b_i) = 1$ in $\mathbf{C}[X]$,
- b_i is unitary,

and such that $Q_0 = 1$.
Let $P_0(X) = \mathrm{ppcm}(b_1, \ldots, b_n)$ and $P = P_0 Q$.
The zeros of the b_i's which are points z of $\mathbf{C}$ where $Q(z, Y)$ is not well defined become the zeros of P_0, points where $P(z, Y)$ is of degree $< n$. Then $\Delta_P = \Delta_Q$, where

$$\Delta_P = \{z \in \mathbf{C} \mid P(z, Y) \text{ has a multiple root or is of degree strictely less than n}\} \ ,$$

$$\Delta_Q = \{z \in \mathbf{C} \mid Q(x, Y) \text{ has a multiple root or is not well defined}\} \ .$$

In the rest of this paper we will speak indifferently of P or Q since it is not difficult to find one from the other.

We claim that S is isomorphic to the Riemann surface $S(P)$ as constructed in the paragraph 2.1, and that we will understand all the meromorphic functions on S through f and g.

Theorem 2.14. *Let S' be $S \setminus f^{-1}(\Delta_P)$. Then the map*

$$\begin{aligned} \Phi \ : \ S' &\longrightarrow S'(P) \\ x &\longmapsto (f(x), g(x)) \ , \end{aligned}$$

is a cover isomorphism. Thus S and $S(P)$ are isomorphic.

Proof. The map Φ is not well defined in all S. This map is well defined in S':
If x is such that $z = f(x) \in \mathbf{C} \setminus \Delta_P$, then the value $\zeta = g(x)$ must be one of the roots of $P(z, Y)$. Indeed $P(f, g) = 0$ and so $\Phi(x) \in S'(P)$ is well defined. The map $f : S' \to \mathbf{C} \setminus \Delta_P$ is a cover (f is meromorphic and we forgot its poles and ramification points) and $\mathbf{z} : S'(P) \to \mathbf{C} \setminus \Delta_P$ is also a cover. We claim that Φ is a cover isomorphism: It is obviously fiber preserving and continuous so it is a cover morphism. Let us take a point $z \in \mathbf{C} - \Delta_P$, such that g separate the points above z. The map

$$\begin{aligned} S' \supset f^{-1}(z) &\to \{\zeta_1, \ldots, \zeta_n\} \ , \\ x &\mapsto (f(x), g(x)) \ , \end{aligned}$$

is a bijection (where ζ_i are the roots of $P(z, Y)$), so $\Phi : S' \to S'(P)$ is a cover isomorphism. Since both covers are analytic, Φ is a biholomorphism from S' to $S'(P)$ which are S and $S(P)$ minus a finite number of points. Thus Φ extends into an isomorphism thanks to proposition 1.6 and we are done. $\square$

We now have

Theorem 2.15. *$\mathcal{M}(S) = \mathbf{C}(f,g)$ and $\mathcal{M}(S(P)) = \mathbf{C}(z,\zeta)$.*

So the meromorphic functions on $S(P)$ are rational functions of z and ζ. We can ask now what do the morphisms between $S(P)$ and $S(Q)$ looks like for $P, Q \in \mathbf{C}[X,Y]$ irreducible polynomials that define two Riemann surfaces. If f and g are meromorphic on $S(P)$ such that $Q(f,g) = 0$ then the mapping $(f,g) : S(P) \to S(Q)$ is well defined and holomorphic.

Proposition 2.16. *Any holomorphic map $\phi : S(P) \to S(Q)$ is equal to*

$$\phi : (z,\zeta) \mapsto (f(z,\zeta), g(z,\zeta)) \,,$$

with $f = \dfrac{a_1}{b_1}$ and $g = \dfrac{a_2}{b_2}$ where $a_1, a_2, b_1, b_2 \in \mathbf{C}[X,Y]$ satisfy :

$$b_1, b_2 \notin (P)$$
$$b_1^n b_2^m Q\left(\frac{a_1}{b_1}, \frac{a_2}{b_2}\right) \in (P) \,,$$

for $n = \mathrm{deg_X Q}$ and $m = \mathrm{deg_Y Q}$.

Proof. Composing ϕ with the two projections on $S(Q)$ we see that $\phi = (f,g)$ with f and g meromorphic on $S(P)$ so on the form $\frac{a_i}{b_i}$, with $b_i \notin (P)$ otherwise they will not be well-define. In order to be valued in $S(Q)$ we need to have

$$Q\left(\frac{a_1}{b_1}, \frac{a_2}{b_2}\right) = 0 \,,$$

on $S(P)$, and by clearing denominators

$$b_1^n b_2^m Q\left(\frac{a_1}{b_1}, \frac{a_2}{b_2}\right) \in (P) \,.$$

□

Proposition 2.17. *Two Riemann surfaces $S(P)$ and $S(Q)$ are isomorphic if and only if there exist some polynomials a_i, b_i, c_i, d_i, e_i with $1 \le i \le 2$ and R, S, all in $\mathbf{C}[X,Y]$ such that*

$$b_i,\ d_i \notin (P) \,,$$
$$b_1^n b_2^m Q\left(\frac{a_1}{b_1}, \frac{a_2}{b_2}\right) = RP \,,$$
$$d_1^{n'} d_2^{m'} P\left(\frac{c_1}{d_1}, \frac{c_2}{d_2}\right) = SQ \,,$$
$$b_1^i b_2^j \left(c_1\left(\frac{a_1}{b_1}, \frac{a_2}{b_2}\right) - X d_1\left(\frac{a_1}{b_1}, \frac{a_2}{b_2}\right)\right) = e_1 P \,,$$
$$b_1^{i'} b_2^{j'} \left(c_2\left(\frac{a_1}{b_1}, \frac{a_2}{b_2}\right) - Y d_2\left(\frac{a_1}{b_1}, \frac{a_2}{b_2}\right)\right) = e_2 P \,.$$

with $n, m, n', m', i, j, i', j'$ are integers to kill denominators of the equations.
The isomorphism is given as above by $\phi = (f,g)$ where $f = a_1/b_1$, $g = a_2/b_2$, and its inverse by $\psi = (h,k) = (c_1/d_1, c_2/d_2)$.

Proof. As one can check (look at [6] part 3.2) these equations are necessary to have an isomorphism. If they hold, then since $\psi \circ \phi = id_{S(P)}$ the two morphisms are isomorphisms. $\square$

To conclude this paragraph, we should have talked here about the morphism part. A pair (S, f) where S compact Riemann surface and $f \in \mathcal{M}'(S)$ gives us a manner to explain $\mathcal{M}(S)$ as a field extension of transcendental degree 1 of $\mathbf{C}$. We can then recover our Riemann surface together with the meromorphic function by taking the Riemann surface $S(P)$ associated to P the minimal polynomial of a primitive elements g in $\mathcal{M}(S)/\mathbf{C}(f)$. We claim that when we take two extensions of $\mathbf{C}(z)$ of transcendental degree 1, $\mathbf{C}(X)[Y]/(P_1)$ and $\mathbf{C}(X)[Y]/(P_2)$, together with a morphism α between them it gives us a morphism of analytic cover between the two pairs $(S(P_1), \mathbf{z})$ and $(S(P_2), \mathbf{z})$. See [4] part 6.2 for the details.

We can then identify as desired :

- Isomorphism classes of Riemann surface together with a non constant meromorphic function,
- Isomorphism classes of ramified covers of the sphere,
- Isomorphism classes of covers of the sphere minus some points,
- Extensions of $\mathbf{C}$ of transcendental degree 1.

And (even if it has not been fully demonstrated) in a functorial way.

3 Examples

Take for example the polynomial $P(X, Y) = Y$. Its solutions on $\mathbf{C}^2$ are $\mathbf{C} \times \{0\}$. Over each point of $\mathbf{C}$ there is only one root in Y (namely 0) and so the cover is a one sheeted cover. So the compactification will add only one point over ∞, with no ramification, and the Riemann surface we get is Σ (remark that it is defined over $\mathbf{Q}$).

An easy example of ramified cover is given by (Σ, f) where $f \in \mathbf{C}[X]$. The map f will ramify at each root of f' and at ∞ with order $d = deg(f)$ as one can see it looking at $z \mapsto 1/f(1/z)$.

For an elliptic curve S (Riemann surface of genus 1), it can be written as the zero set of

$$P(X, Y) = Y^2 - X(X - 1)(X - \lambda) .$$

with $\lambda \in \mathbf{C} \setminus \{0, 1\}$. To see that, (See [11] § 5.7) take S as a quotient of $\mathbf{C}$ by a lattice generated by 1 and $\tau \in \mathbb{H}$. Using Weierstrass modular form one can write S as the zero set of

$$Y^2 - X^3 + g_2 X + g_3 ,$$

with g_2 and g_3 some numbers. By an affine changing of coordinates we find the polynomial P.

The projection $\mathbf{z}$ in the first coordinate will ramify at (z, ζ) when $\partial_Y P(z, \zeta)$ vanish together with P. But $\partial_Y P(X, Y) = 2Y$ implies $\zeta = 0$ and z must be $0, 1, \lambda$ or ∞.[2] The

[2] We know that it ramifies over ∞ because we can write the polynomial in another chart $X^2 Y^2 P(1/X, 1/Y)$ and it will vanish in $(0, 0)$ of order 2. See next §.

cover is two sheeted, one can see this looking at a regular point : if $z \notin \{0, 1, \lambda, \infty\}$ then $P(z, Y) = Y^2 - z(z-1)(z-\lambda)$ has two different roots. It admits an isomorphism that is an involution $(z, \zeta) \mapsto (z, -\zeta)$.

This example can be generalized, we define a *hyperelliptic curve* as being a Riemann surface admitting a degree two meromorphic function. We can write this surface as the zero set of

$$P(X, Y) = Y^2 - (X - \lambda_1)(X - \lambda_2) \cdots (X - \lambda_n) .$$

The map $\mathbf{z}$ ramifies over ∞ only if n is odd, so there is always an even number of ramifications. To see this write

$$X^n Y^2 P(1/X, 1/Y) = X^n - Y^2(1 - \lambda_1 X) \cdots (1 - \lambda_n X) .$$

Since near $(0, 0)$ the map $f : z \mapsto (1 - \lambda_1 z) \cdots (1 - \lambda_n z)$ is holomorphic and does not vanish we can write in a neighbourhood of $(0, 0)$

$$\zeta^2 = \frac{z^n}{g(z)^n} ,$$

where $g = f^{1/n}$. Now we see that we can find a holomorphic map h with $\zeta^2 = zh^2(z)$ if n is odd or $\zeta^2 = h^2(z)$ if n is even. In the first case, ζ must be like $\sqrt{z}h(z)$ so it ramifies (and just one point over ∞) but in the second $\zeta = \pm h$ (and two points over ∞).
To simplify the setting we say that $n = 2k$ is even. Using the Riemann Hurwitz formula (theorem 1.7) we find that

$$2 - 2g = 2 * 2 - n ,$$

since each ramification point is of order 2 and they are n. So the genus of S is $g = \frac{n-2}{2} = k - 1$ and $n = 2g + 2$. The covering group of an hyperelliptic surface together with $\mathbf{z}$ is $\{\mathrm{id}, j\}$ where the involution j is $j(z, \zeta) = (z, -\zeta)$.

4 Belyi's theorem

We prove here an important result which is at the beginning of the theory of "dessins d'enfants".

Definition 4.1. *We call (S, f) a* Belyi pair *if S is a Riemann surface and $f : S \to \Sigma$ is a ramified cover of Σ ramified over $\{0, 1, \infty\}$. The map f is called a* Belyi map. *In the case $S \simeq \Sigma$ and f is a polynomial, then f is called a* Belyi polynomial *(in fact it is a polynomial with branch values 0 and 1).*

Theorem 4.2 (Belyi)**.** *The following are equivalents :*

(i) S is an algebraic curve defined over $\overline{\mathbf{Q}}$.

(ii) There exists $f \in \mathcal{M}'(S)$ branched over $\Delta \subset \{0, 1, \infty\}$.

So a Belyi pair is defined over $\overline{\mathbf{Q}}$. A Belyi map is a rational function in $\mathbf{C}(f, g)$, and at the end of the proof we will see that it can always be written with coefficients in $\overline{\mathbf{Q}}$.

4.1 The first part

Let's start with an example. Look at the elliptic curve defined by the polynomial

$$P(X, Y) = Y^2 - X(X - 1)(X - \lambda) ,$$

where $\lambda = \frac{p}{p+q} \in \mathbf{Q}$ and $p, q \in \mathbf{N}^*$. We want to find a Belyi map defined on it (it is defined over $\overline{\mathbf{Q}}$). Starting with $f = \mathbf{z}$, that ramifies over $\{0, 1, \lambda, \infty\}$, we will find it. Let us study the polynomial

$$R(X) = X^p(1 - X)^q .$$

It represents a meromorphic function of the sphere. Since

$$R'(X) = X^{p-1}(1 - X)^{q-1}(p(1 - X) - qX) = X^{p-1}(1 - X)^{q-1}(p - (p + q)X) ,$$

the map R' vanish at λ and maybe at 0 and 1. Thus the ramification values of R are $\{R(0), R(1), R(\lambda)\} = \{0, R(\lambda)\}$. Then the composition $g = R \circ f$ gives us a new meromorphic function, and by proposition 1.2 we know that $\Delta_g \subset \{0, R(\lambda), \infty\}$, so only 3 ramification values. By taking $R/R(\lambda)$ instead of R, the ramification values of g are in $\{0, 1, \infty\}$ and $(S(P), g)$ is a Belyi pair.

Proof of the first part. Let suppose that S is defined over $\overline{\mathbf{Q}}$.

First step, let us show that there exist meromorphic functions branched over $\overline{\mathbf{Q}}$.
We can write $S \simeq S(P)$ with $P \in \overline{\mathbf{Q}}[X, Y]$ irreducible of degree n in Y. Let say $P(X, Y) = P_n(X) + \ldots + P_0(X)Y^n$. The meromorphic functions on S are elements of $\mathbf{C}(X)[Y]/(P)$. Let's take a non constant meromorphic function g in $\overline{\mathbf{Q}}(X)[Y]/(P)$, for example we will take $g = \mathbf{z}$ corresponding to the polynomial X but it works with any $f \in \overline{\mathbf{Q}}(X)[Y]/(P)$ (look at [4] part 7.2.2). g must be ramified over $\overline{\mathbf{Q}} \cup \{\infty\}$ because if $x \in S$ is a ramification point then $\mathbf{z}(x)$ must lie in $\Delta(P) \cup \{\infty\}$ and $\Delta(P)$ are elements of $\mathbf{C}$ that vanish some polynomial (namely the resultant $Res_Y(P, \partial_Y P) \in \overline{\mathbf{Q}}[X]$) defined over $\overline{\mathbf{Q}}$ so are algebraic.

Second step, let us show that there exist meromorphic functions ramified above $\mathbf{Q} \cup \{\infty\}$.
Let Δ be the (finite) set of branch values of g. Let μ_1 be the minimal polynomial over $\mathbf{Q}$ of the non-rational elements of Δ. So $\Delta \subset \mathbf{Q} \cup \{\infty\} \cup \mu_1^{-1}(0)$ with $\mu_1 \in \overline{\mathbf{Q}}[x]$ of degree d_1. Let us look at $g_1 = \mu_1 \circ g$. The finite ramification values of g_1 are :

- The images by μ_1 of the ramification values of g, which are rational (if the value is rational), and zero (if the value is not rational),
- The ramification values of μ_1.

But μ_1' is a polynomial of degree $d_1 - 1$ so there is at most $d_1 - 1$ non-rational values. Since $\mu_1' \in \mathbf{Q}[x]$, the minimal polynomial of its roots is of degree less than or equal to $d_1 - 1$.
Let define μ_2 as the minimal polynomial of the non-rational ramification values of g_1. The set of ramification values of g_1 will be denoted by Δ_1. We have $\Delta_1 \subset \mathbf{Q} \cup \{\infty\} \cup \mu_2^{-1}(0)$ with $\mu_2 \in \mathbf{Q}[x]$ of degree $d_2 \leq d_1 - 1$.
Let us continue until μ_m is of degree 1. So the ramification values Δ_m of $g_m = \mu_m \circ \ldots \circ \mu_1 \circ g$ are infinite or rational.
So for the rest of the proof we note g for a non-constant meromorphic function on S

with ramification values $\Delta \subset \mathbf{Q} \cup \{\infty\}$.

Last step, let us show that there exist meromorphic functions with only 3 branch values.
Let suppose that Δ contains three different finite values $x_1 < x_2 < x_3$. Let α be the affine function that maps x_1 to 0 and x_3 to 1, so $0 < \alpha(x_2) < 1$. We can write $\alpha(x_2) = \frac{p}{p+q}$ where p and q are positive integers.
Define $R(x) = x^p(1-x)^q$ as in the example above. R' vanish on $\frac{p}{p+q}$ and eventually in 0 and 1, and α does not ramify, so the branch values of $R \circ \alpha \circ g$ are

$$\{R(0) = R(1) = 0, R(\frac{p}{p+q})\} \cup R \circ \alpha(\Delta \setminus \{x_1, x_2, x_3\}) \,,$$

and there is at least one less element than in Δ.
So we can by composing with polynomials reduce the number of values to 3, which we can suppose to be $\{0, 1, \infty\}$. This ends the first part of the proof. $\square$

4.2 The second part

The second part is much complicated (even if it came first) and we will discuss a little the Galois action and finite ramified covers of the sphere before prove it.
Ramified covers of the sphere over n points are not interesting until $n = 3$. Let's think about connected ramified covers of the sphere with less than three branch values :

- If a cover doesn't have branch values it must be a biholomorphism (the sphere is simply connected so all the covers of it are trivial).
- If a cover $p : S \to \Sigma$ has only one branch value, we can suppose that is is ∞, and so $p : S \setminus p^{-1}(\{\infty\}) \to \mathbf{C}$ is a cover and since $\mathbf{C}$ is simply connected, it must be a biholomorphism and so $S \simeq \Sigma$.
- If a cover $p : S \to \Sigma$ has two branch values (let's suppose that they are 0 and ∞) then $p : S \setminus p^{-1}(\{0, \infty\}) \to \mathbf{C}^*$ is a connected cover so must be isomorphic to $z \mapsto z^d$ from $\mathbf{C}^*$ to $\mathbf{C}^*$ and $S \simeq \Sigma$ again.

But the case of three branch values is much more complex and we will find a combinatorial way to represent them : the dessins d'enfants.

Action of Galois group and the first invariant

Given a Belyi pair (S, p), since we can suppose that $S = S(P)$ and $p \in \mathbf{C}(z, \zeta)$, we can look at the action of an element $\sigma \in \mathrm{Gal}(\mathbf{C})$ on the pair which will be obviously defined by acting on the coefficients of P and p. Then we will obtain another polynomial which define a Riemann surface P^σ together with a rational map p^σ. A point $x \in S$ is a ramification point of order n if $p'(x) = \ldots = p^{(n-1)}(x) = 0$ and $p^{(n)}(x) \neq 0$. These conditions can be written in terms of vanishing some polynomials, and non-vanishing others, but these conditions are σ-invariant. Moreover, since $\{0, 1, \infty\}$ is obviously σ-invariant too, p^σ is a Belyi map and it has the same valency as p.

Definition 4.3. *The* valency *of a Belyi map is a triple of sets, respectively the set of ramification orders above 0, 1 and ∞.*

Remark 4.4. *Let $p : S \to \Sigma$ be a Belyi map with valency*

$$(\{n_{0,1}, \dots, n_{0,k_0}\}, \{n_{1,1}, \dots, n_{1,k_1}\}, \{n_{\infty,1}, \dots, n_{\infty,k_\infty}\}) \ .$$

Thanks to Riemann Hurwitz formula (theorem 1.7) we can compute the genus of S :

$$2g - 2 = n - k_0 - k_1 - k_\infty \ .$$

End of the proof of Belyi's theorem

Recall that (see section 1.2) we know that a Belyi pair is uniquely determined by it's monodromy. It will be useful.
We found this proof in [6]. You will find details about specialization just after the proof, and they can be found in [6] too.

Proof of the second part of Belyi's theorem. Since the valency is invariant under the action, we have only finitely many different monodromies possibles. Thus (since a cover is determined up to isomorphisms by its monodromy) the orbit contains only finitely many different Riemann surface classes.
This will be enough to have that S is defined over $\overline{\mathbf{Q}}$.

We write S as $S(P)$. Let us look at the field generated by the coefficients of P. It should be written as $\mathbf{Q}(x_1, \dots, x_k, \alpha)$ with $\{x_1, \dots, x_k\}$ algebraically independents and α algebraic over $\mathbf{Q}(x_1, \dots, x_k)$ of minimal polynomial μ_α.
If $\sigma \in \mathrm{Gal}(\mathbf{C})$, then the field of the coefficients of P^σ is $\mathbf{Q}(\sigma(x_1), \dots, \sigma(x_k), \sigma(\alpha))$. By finiteness of the isomorphism classes under the $\mathrm{Gal}(\mathbf{C})$ action, and uncountableness of $\mathrm{Gal}(\mathbf{C})$ (there is an uncountable number of transcendental numbers), there should be an uncountable number of $\xi, \eta \in \mathrm{Gal}(\mathbf{C})$ such that $S^\xi \simeq S^\eta$ and we can choose $\sigma = \xi^{-1}\eta$ such that $\{x_1, \dots, x_k, \sigma(x_1), \dots, \sigma(x_k)\}$ is an algebraically independent set ($\mathbf{Q}[x_1, \dots, x_n]$ is countable). There is then an isomorphism

$$\Phi := \Psi^{\xi^{-1}} : S = S(P) \to S(P^\sigma) = S^\sigma \ ,$$

where $\Psi : S^\xi \to S^\eta$ is an isomorphism. Let's note $x_{k+l} := \sigma(x_l)$ for all $1 \leq l \leq k$. The fact that Φ is an isomorphism implies the existence of several polynomials satisfying some conditions (see proposition 2.17), so looking at their coefficients we can take a maximal set of algebraically independent elements $\{x_{2k+1}, \dots, x_m\}$ that are algebraically independent of the x_i ($1 \leq i \leq 2k$). So the field generated by the coefficients of P, P^σ and all the polynomials defining Φ can be written $\mathbf{Q}(x_1, \dots, x_m, \beta)$ where β is algebraic over $\mathbf{Q}(x_1, \dots, x_m)$. Let's chose in $\mathbf{Q}[i]$ some numbers $q_{k+1}, \dots, q_m$ such that $(x_1, \dots, x_k, q_{k+1}, \dots, q_m)$ is an infinitesimal specialization of $(x_1, \dots, x_m, \beta)$ ($\mathbf{Q}[i]$ is dense in $\mathbf{C}$) with the associated morphism $\mathbf{s} : \mathbf{Q}[x_1, \dots, x_m, \beta] \to \mathbf{C}$. Let's look at the elements of $\mathbf{Q}(x_1, \dots, x_m, \beta)$ that are well defined through $\mathbf{s}$ (i.e. some R/S with $R, S \in \mathbf{Q}[X_1, \dots, X_{m+1}]$ such that $S(x_1, \dots, x_k, q_{k+1}, \dots, q_m, \beta)$ does not vanish). The q_i's can be chosen such that the image of the coefficients of all the polynomials[3] defining Φ by $\mathbf{s}$ are well defined. Then we can apply $\mathbf{s}$ to the polynomial identities that makes Φ an isomorphism (prop 2.17), to show that

$$\mathbf{s}(\Phi) : S(\mathbf{s}(P)) \to S(\mathbf{s}(P^\sigma)) \ ,$$

[3] These coefficients are elements of $\mathbf{Q}(x_1, \dots, x_m, \beta)$.

is an isomorphism. Since it is an infinitesimal specialization, we have that $\mathbf{s}(P) = P$ so we have an isomorphism from P to $\mathbf{s}(P^\sigma)$. But we have chosen the coefficients of the specialisation such that the coefficients of $\mathbf{s}(P^\sigma)$ lie in $\mathbf{Q}[i] \cup \{\mathbf{s}(\beta)\}$ so are algebraic. And the proof is done : we have an isomorphism between $S(P)$ and a Riemann surface defined over $\overline{\mathbf{Q}}$.

□

The same kind of proof can be done to show that Belyi maps are defined over $\overline{\mathbf{Q}}$ too, i.e. there exists some isomorphism fiber-preserving

$$\begin{tikzcd} S(P) \arrow[rr, "\Phi"] \arrow[dr, "p"'] & {} \arrow[d, phantom, "\circlearrowleft"] & S(Q) \arrow[dl, "q"] \\ & \Sigma & \end{tikzcd}$$

with the coefficients of Q **and** q are in $\overline{\mathbf{Q}}$. For this see [6] part 3.8.

Definition 4.5. *Given an algebraic independent set $\{x_1, \ldots, x_k\} \subset \mathbf{C}$, and given k complex numbers (y_i), the map sending x_i to y_i gives us a well defined morphism from $\mathbf{Q}[x_1, \ldots, x_k]$ to $\mathbf{C}$. We call the y_i's together with the morphism ϕ a* specialization *of the set $\{x_1, \ldots, x_k\}$ and its* distance *will be the number $max_i|x_i - y_i|$.*

Proposition 4.6. *Given an algebraic independent set $\{x_1, \ldots, x_k\}$ and α algebraic over $\mathbf{Q}(x_1, \ldots, x_k)$, given k complex numbers $\{y_1, \ldots, y_k\}$, we note ϕ the morphism of specialisation of the x_i's and $\mu \in \mathbf{Q}(x_1, \ldots, x_k)[X]$ the minimal polynomial of α. By clearing denominators of the coefficients of μ we can write it as a polynomial in $\mathbf{Q}[x_1, \ldots, x_k][X]$. So the morphism of specialization send μ to a polynomial $\phi(\mu)$. We claim that to be a morphism, a map from $\mathbf{Q}(x_1, \ldots, x_k, \alpha)$ to $\mathbf{C}$ extending ϕ needs to send α on a root of the polynomial $\phi(\mu)$. Conversely, by sending α on a root of $\phi(\mu)$, we define a morphism $\psi : \mathbf{Q}(x_1, \ldots, x_k, \alpha) \to \mathbf{C}$ extending ϕ.*

The proof need just an algebraic verification.

Proposition 4.7. *Given an algebraic independent set $\{x_1, \ldots, x_k\}$ and α algebraic over $\mathbf{Q}(x_1, \ldots, x_k)$ of minimal polynomial μ, let's set $d = min_{j \neq k}|\alpha_j - \alpha_k|$ with $\{\alpha_j\}_j$ the roots of μ. Then there exists $\varepsilon > 0$ such that for every specialization $\phi : x_i \mapsto y_i$ of distance less than ε, there is a unique root β of $\phi(\mu)$ such that $|\alpha - \beta| < d/2$.*

Proof. Say that ϕ is a specialization of distance ε. If ε become small, then the coefficients of $\phi(\mu)$ are close to those of μ and since the roots depend continuously of the coefficients, there will be a ε such that each root β_j of $\phi(\mu)$ verify $|\beta_j - \alpha_j| < d/2$ for some α_j root of μ, so choose β corresponding to α. Then the uniqueness comes easily, because for such a ε, if $|\beta_j - \alpha| < d/2$ then $|\alpha - \alpha_j| \leq |\alpha - \beta_j| + |\alpha_j - \beta_j| < d$ (with α_j corresponding to β_j) so by definition of d we must have $\alpha = \alpha_j$ so $\beta = \beta_j$ because β_j must be at distance at least $d/2$ of each other α'_j than α. □

Definition 4.8. *With the same notations as in the proposition above, we will say that $(y_1, \ldots, y_n, \beta)$ of distance less than ε is an* infinitesimal specialization.

5 Dessins d'enfant

We start this section by another point of view for the Belyi pairs.

5.1 Triangle groups and Belyi pairs

For most proofs about tesselations and triangle groups see [4] chapter 6.
Let's take (S, f) a Belyi pair.
$\mathbb{H}$ will be the Poincaré disc and $\mathfrak{h}^+$ will designate the upper half plane in Σ. Take a triangle Δ_1 in $\mathbb{H}$ with vertices a, b, c and with angles $\pi/p, \pi/q, \pi/r$ respectively, with $p, q, r \in \mathbf{N}$ well chosen (common multiples of the orders of ramification of f in $0, 1, \infty$ respectively). This triangle gives us a tessellation of $\mathbb{H}$:
Let us denote by G the group of isometries of $\mathbb{H}$ that are generated by the reflexions along the sides of Δ_1. A fundamental domain of this group is the triangle Δ_1. Let $j : \Delta_1 \to \mathfrak{h}^+$ be the uniformization of the triangle, where the upper half plane is seen in Σ, such that a (respectively b, c) maps to 0 (respectively $1, \infty$). One can continue j through $\mathbb{H}$:
Let us look at the triangle Δ_1, and at his reflexion $\sigma(\Delta_1) = \Delta_2$ by one side, the geodesic γ passing through a and b (for example). The boundary of the triangle Δ_1 is send to $\mathbf{R}$ by j and the interior to the upper half plane in Σ. We will continue the function to the second triangle by reflexion. We can suppose that γ is $[-1, 1]$ and a is 0 (using a homography) to be in this situation :

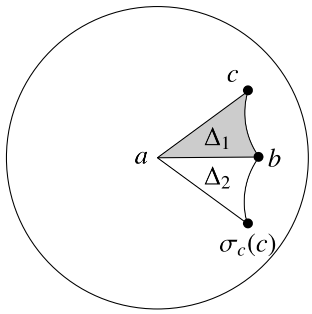

Figure 1: Action of σ on Δ_1.

If $x' \in \Delta_2$, it is the symmetric of a unique $x \in \Delta_1$ and we can define $j(x') = \overline{j(x)}$. So Δ_2 is sent to the lower half plane. The function j defined is holomorphic in Δ_2 since the reflexion σ is antiholomorphic and the conjugation too. So j is holomorphic in $\Delta_1 \cup \Delta_2$ and continuous in $]a, b[$ so it is holomorphic in $\Delta_1 \cup \Delta_2 \cup]a, b[$.[4]
We define therefore a well defined function $j : \mathbb{H} \to \Sigma$ that is holomorphic in every point of $\mathbb{H}$ except maybe at the images of the vertices a, b and c by G (the vertices of the tessellation). These points are sent to $0, 1, \infty$ respectively and j is bounded near these points so j is holomorphic in $\mathbb{H}$. In fact j ramifies at the vertices of the tessellation, this can be seen by looking for example at a. Take a little neighbourhood U around a. The tessellation gives us $2p$ triangles at a : $\Delta_1, \ldots, \Delta_{2p}$. Each $\Delta_k \cap U$ is sent to $V \cap \mathfrak{h}^+$ if k is odd or to $V \cap \mathfrak{h}^-$ if k is even where $V = j(U)$, so a is a zero of order p.

We will work now with G_+ which is just the subgroup of the elements in G that are holomorphic. In fact G_+ is generated by the rotations[5] ρ_a of angle $2\pi/p$ around a,

[4] To prove this we can use Morera theorem.
[5] For example ρ_a is just $\sigma_c \circ \sigma_b$ where σ_x is the reflexion along the segment at the opposite vertex of x in Δ_1.

ρ_b of angle $2\pi/q$ around b and ρ_c of angle $2\pi/r$ around c. A fundamental domain for G_+ can be $\Delta_1 \cup \Delta_2$, j is invariant under the action of G_+ and pass to the quotient to an isomorphism $j : \mathbb{H}/G_+ \to \Sigma$.

We have two holomorphic maps :

$$\begin{tikzcd} & S \arrow[d, "f"] \\ \mathbb{H} \arrow[r, "j"'] & \Sigma \end{tikzcd}$$

where the vertical one is $f : S \to \Sigma$ our meromorphic function ramified over $\{0, 1, \infty\}$. If $x \in \mathbb{H}$ is not a vertex of the tessellation, then there exists a neighbourhood U of x such that $f^{-1} \circ j$ is well define on U (and is biholomorphic, in fact j and f are local biholomorphisms except at the ramification points) up to chose an element $y \in f^{-1}(j(x))$. Since p (respectively q and r) is common multiple of the orders of ramification of f in 0 (respectively in 1, ∞) we have that that $f^{-1} \circ j$ can be continued along every path of $\mathbb{H}$ and by monodromy theorem (see [5]) there exist a globally defined holomorphic map $g : \mathbb{H} \to S$ that is equal to $f^{-1} \circ j$ in U. The map g will be equal to $f^{-1} \circ j$ on every U' that is sent by j to a simply connected set in Σ. Thus g is a (possibly ramified) cover of S which is G_+-equivariant, and $S \simeq \mathbb{H}/\Gamma$ where $\Gamma = \{\gamma \in G_+ \mid g(\gamma z) = g(z)$ for every $z \in \mathbb{H}\}$ is a subgroup of G_+. Be careful, in general Γ is not a surface group because it has elements of finite order and its action is not free.

So we can write our Belyi pair as the projection

$$\mathbb{H}/\Gamma \to \mathbb{H}/G_+ \, .$$

The fibers can be identify with G_+/Γ and the monodromy of our Belyi pair can be written as the action of G_+ on G_+/Γ by left multiplication.

5.2 Dessins

Following L. Schneps in [1], we define the "dessins d'enfants" as follows.

Definition 5.1. - *A* dessin d'enfant *is a quadruple (S, E, V, c) where S is the topological model of a Riemann surface, $V \subset E$ is a finite set of points, the vertices, $E \subset S$ is such that $E - V$ consists of finitely many disjoints "1-cells"[6], the edges, $S - E$ consists of a finite number of "2-cells"[7], the faces, and c is a bipartite structure on the set of vertices V, i.e. a map $c : V \to \{\circ, \bullet\}$ such that if two vertices v_1, v_2 are joined by an edge we have $c(v_1) \neq c(v_2)$.*

- *Two dessins (S, E, V) and (S', E', V') are said* isomorphic *if there exist a homeomorphism Φ from S to S' inducing homeomorphisms from E to E' and from V to V'.*

- *An isomorphism class of dessins will be called an* abstract dessin.

Proposition 5.2. *Let's take (S, p) a Belyi pair. The topological space underlying S together with the graph $p^{-1}([0, 1])$ with white vertices $p^{-1}(\{0\})$ and black vertices $p^{-1}(\{1\})$ is a dessin d'enfant.*

[6]Subsets of S that are homeomorphic to segments.
[7]Subsets of S that are homeomorphic to discs.

Proof. First, since $p_{|\Sigma\setminus\{0,1,\infty\}}$ is a cover and $]0,1[$ is simply connected, $p^{-1}(]0,1[)$ is a disjoint union of 1-cells. Every of these cells join a white vertex to a black one since $]0,1[$ join 0 and 1. Then, $p : S \setminus p^{-1}([0,1]) \to \Sigma \setminus [0,1]$ is a ramified cover ramified above only one point (∞) so each connected component of $S \setminus p^{-1}([0,1])$ must be homeomorphic to a disc, i.e. is a 2-cell. □

In fact, a Belyi pair (S, p) gives us a bicolored triangulation of our surface :

Definition 5.3. *A* bicolored triangulation *of the topological model S of a Riemann surface is the following data :*

- *A finite set $V \subset S$ of* vertices $\{V_i\}_i$*,*
- *A set $E \subset S$ of* edges $\{E_i\}_i$*,*

satisfying the following conditions :

- *$E - V = \sqcup e_i$ is a finite disjoint union of 1-cells with $\overline{e_i} = e_i \sqcup v_{i_1} \sqcup v_{i_2} = E_i$,*
- *$S - E = \sqcup T_i$ is a finite disjoint union of 2-cells (the* faces/triangles*) such that for each triangle T, its boundary $\partial T = e_{i_1} \sqcup e_{i_2} \sqcup e_{i_3} \sqcup v_{j_1} \sqcup v_{j_2} \sqcup v_{j_3}$ consists of three distinct vertices joined by three distinct edges where two different edges in this boundary have one and only one common vertex,*
- *There is a tripartite structure on the set of vertices, namely $c : V \to \{\circ, \bullet, \times\}$ such that if two vertices V_1, V_2 are joined by an edge, then $c(V_1) \neq c(V_2)$ and if three vertices V_1, V_2, V_3 are in the boundary ∂T of a triangle T then $c(V_i) \neq c(V_j)$ for all $i \neq j$.*

The terminology "bicolored triangulation" is used because there is two kind of triangles : the orientation of S gives us an order of the vertices of a triangle and so there is two types of triangles since there is only two different order in the set of three elements $\{\circ, \bullet, \times\}$. The gray triangles will be those of oriented boundary $(\circ - \bullet - \times)$ and the white ones with oriented boundary $(\bullet - \circ - \times)$. Two adjacent triangles are of different colors. Here our meaning of triangulation is not the usual one : two triangles can meet at more than one edge as one can see in the example of the sphere (figure 2).

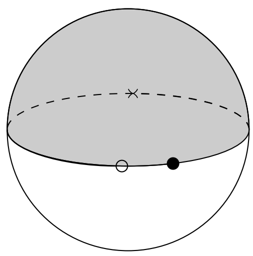

Figure 2: Triangulation of the sphere.

The sphere can be seen triangulated by two triangles, the upper half plane (coloured in grey) and the lower (coloured in white), their boundary are $[0,1] \cup [1,\infty] \cup [\infty,0]$. When pulling-back just $[0,1]$ by p to get the dessin d'enfant one can pull-back the whole triangulation. Since p is a ramified cover, ramified over $\{0, 1, \infty\}$, and since the vertices of the triangulation of Σ are in this set, the pull-back of the triangulation by p is always a triangulation (of S). We mark the "zero" points by $\circ$, the "one" points by $\bullet$ and the "poles" by $\times$ (look at figure 2). The triangles are coloured like in the sphere : if a triangle is a preimage of the upper half plane it is coloured in gray, and in white if it is a preimage of the lower half plane. One can check that the oriented boundary of a triangle is $(\circ - \bullet - \times)$ for a gray triangle, or is $(\bullet - \circ - \times)$ for a white one. This is a bicolored triangulation of the sphere.

Inspired by what happens with a dessin coming from a Belyi pair, taking a dessin d'enfant (S, E, V) we define a triangulation of S by adding a new vertex in each face (in each connected component of $S - E$) and for each new vertex new edges between this point and all the vertices in the boundary of the face (and such that these new edges are all disjoint from the others). To do that one can for example take the face F with n vertices in the boundary, and an homeomorphism $\varphi : \overline{F} \to \overline{\mathbb{D}}$ sending each vertex to a n^{th} root of unity ζ_i. Then in the chart φ the new vertex is 0 and the new edges are the segments $[0, \zeta_i]$. Finally pull-back by φ.

Definition 5.4. *Such a triangulation (S, E', V') is said to be derived from the dessin (S, E, V).*

Proposition 5.5. *The derived triangulations of a dessin are all homotopic relatively to the set E, that is homotopic classes of triangulations derived from a dessin only depends on the abstract dessin.*

Then we can stand for "Grothendieck's correspondence" :

Theorem 5.6. *Abstract dessins are in one-to-one correspondence with isomorphism classes of Belyi pairs.*

Proof. Take two isomorphic Belyi pairs (S, p) and (S', p'), with their associated dessin. The isomorphism seen just as a homeomorphism between topological surfaces will send the first dessin to the second one as an isomorphism of dessin.
On the other hand, given an abstract dessin one can chose a concrete dessin $D = (S, E, V, c)$ in this class, fix a derived triangulation of this dessin, and then construct a Belyi map p (topologically) as follows :
The map p must send homeomorphically each triangle to $\mathfrak{h}^+$ (a gray one) or $\mathfrak{h}^-$ (a white one) with $\circ$-vertices sent to 0, $\bullet$-vertices sent to 1 and $\times$-vertices sent to ∞. It is a ramified cover from S to Σ.
Then we just ask the application p to be holomorphic. There is a unique complex structure on S doing that thanks to theorem 1.14. Taking another concrete dessin D' and making the same construction, a homeomorphism between S and S' that is a dessin isomorphism will give us a biholomorphism of $S \to S'$ by unicity of the good complex structure. □

5.3 Monodromy group on a dessin

One can find the same idea in [1] with more details, but in the case of pre-clean dessins (that is dessins with $\bullet$ valencies less than or equal to 2). So, inspired by example of triangle groups, we define the cartographical group as being the group of "reflexions" along the sides of the triangulation.

Definition 5.7. *The* cartographical group C *is the free group generated by three elements* $\sigma_\circ$, $\sigma_\bullet$, $\sigma_\times$ *with the relations* ${\sigma_\circ}^2 = {\sigma_\bullet}^2 = {\sigma_\times}^2 = e$. *The* oriented cartographical group C_+ *is the subgroup of* C *of index two consisting of even words. The group* C_+ *can be generated by* $\rho_\circ = \sigma_\bullet\sigma_\times$, $\rho_\bullet = \sigma_\times\sigma_\circ$ *and* $\rho_\times = \sigma_\circ\sigma_\bullet$ *(which satisfy the relation* $\rho_\circ\rho_\bullet\rho_\times = e$*).*

The group C acts on a bicolored triangulation of S in the following way :
Let us denote by $F = \sqcup T_i$ the (finite) set of triangles. We only need to define the action of the generators of C. Given a triangle, let say T with its vertices $\{\bullet, \circ, \times\}$, we have three other triangles (not all different in general but different from T_1) let say $T_\circ$ with common edge $(\bullet - \times)$, $T_\bullet$ with common edge $(\times - \circ)$ and $T_\times$ with common edge $(\circ - \bullet)$. Then the action is easily defined by

$$\sigma_\circ \cdot T := T_\circ\ , \quad \sigma_\bullet \cdot T := T_\bullet\ , \quad \sigma_\times \cdot T := T_\times\ .$$

This action is transitive on the set of triangles.
The restriction of the action to C_+ gives us two orbits : the white triangles and the gray.

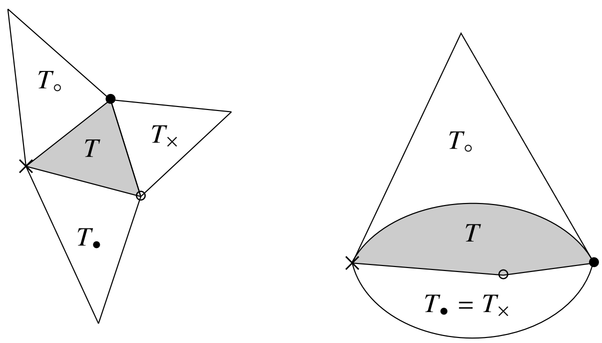

Figure 3: Examples of a triangle T and its adjacent triangles.

Take a triangle $T \in F$. Such a triangle has only one edge $(\circ - \bullet)$. Looking at its orbit by the action of C_+ (which must be finite since there is a finite number of triangles) one can ask what is the stabilizer H of it. We can then identify the orbit with C_+/H. The stabilizer H is a finite index subgroup of C_+, and moreover if we take another triangle $\rho T = T'$ in the orbit of T, then the stabilizer of T' will be $H' = \rho H \rho^{-1}$. So an abstract dessin comes together with a conjugation class of subgroup of finite order of C_+. To recover a dessin from a subgroup H of C_+ of finite index one can proceed as follows :
We define $L := C_+/H$ as the set of edges of the dessin. The three elements $\rho_\bullet$, $\rho_\circ$ and $\rho_\times$ act on the set L by left multiplication. The orbits $L/\langle\rho_\circ\rangle$ are the $\circ$-points, and the orbits $L/\langle\rho_\bullet\rangle$ are the $\bullet$-points. We can then draw the dessin D as a bipartite graph with a cyclic ordering in each vertex. One can calculate the valency of the dessin, and so the genus of the topological surface to draw the dessin on it. A conjugate of H gives us an isomorphic dessin. Indeed if $H' = \rho H \rho^{-1}$ and D' the associated dessin, we have the same number of edges, $\circ$-points and $\bullet$-points. The map

$$L \to L'\ ,$$

$$e \mapsto \rho e\ ,$$

is an isomorphism.

So we claim that the isomorphism classes of subgroups of C_+ of finite index are in bijection with the abstract dessins.
Then we recover the Grothendieck's correspondence since C_+ is trivially isomorphic to $\pi_1(\Sigma - \{0, 1, \infty\})$, and the monodromy of the cover is encoded by the action of $\rho_\circ$ and $\rho_\bullet$.

5.4 The action of Gal($\overline{\mathbf{Q}}/\mathbf{Q}$)

Let define G =Gal($\overline{\mathbf{Q}}/\mathbf{Q}$). This group acts on the set of abstract dessins since it acts (thanks to Belyi's theorem) on the isomorphism classes of Belyi pairs and thanks to Grothendieck's correspondence. This action may learn us some important setting about G, because the action is faithful. One of the goals of the theory of dessins d'enfants is to find a total collection of invariant but we are not there yet. In this section we will talk about the faithfulness of the action and about some invariants.

Theorem 5.8. *The action is faithful on genus 1 dessins so is faithful on all the dessins.*

Proof. Take a genus 1 dessin. The underlying curve S can be written as the zero set of

$$Y^2 - X(X-1)(X-\lambda) ,$$

with $\lambda \in \mathbf{C} - \{0, 1\}$. It is isomorphic to another curve if and only if they have the same j invariant that will be $j(\lambda) = \frac{4(1-\lambda+\lambda^2)^3}{27(\lambda^2(1-\lambda)^2)}$. The curve is defined over $\overline{\mathbf{Q}}$ if and only if $\lambda \in \overline{\mathbf{Q}}$. Take an element σ of G that send $j(\lambda)$ to an algebraic number different from $j(\lambda)$. The curve S^σ is not isomorphic to S so the two dessins corresponding must be different. □

Definition 5.9. *We call a dessin (S, E, V, c) a* tree *when $S \simeq \Sigma$ and E is simply connected.*

What is interesting is that the action of G is faithful on particular subclasses of dessins. We have seen that it is faithful on the genus one dessins. It is faithful too on the set of genus g dessins, for all g, even in the set of trees.
For the trees you can find the proof in [1] and it is written in french in [4]. For the genus $g > 1$ see [6] part 4.5.3 using hyperelliptic surfaces.

Let us show a characterization of trees, and some examples.

Lemma 5.10. *Given a Belyi pair (S, f) and its associated dessin, then the dessin $E = f^{-1}([0, 1])$ is a retract by deformation of $S \setminus f^{-1}(\{\infty\})$.*

Proof. We can write down a retraction by deformation from $\mathbf{C} = \Sigma \setminus \{\infty\}$ to $[0, 1]$ with some good properties to pull it back to S :

$$h : [0,1] \times \mathbf{C} \to \mathbf{C} ,$$
$$(t, z) \mapsto \begin{cases} tz & \text{if } y \leq 0 \\ x + ity & \text{if } 0 \leq y \leq 1 \\ 1 + tre^{i\theta} & \text{if } y \geq 1 \end{cases} .$$

where $z = x + iy = 1 + re^{i\theta}$, is a homotopy between the retraction ρ and $id_{\mathbf{C}}$ where

$$\rho : \mathbf{C} \to \mathbf{C}\,,$$
$$z = x + iy \mapsto \begin{cases} 0 & \text{if } y \leq 0 \\ y & \text{if } 0 \leq y \leq 1 \\ 1 & if 1 \leq y \end{cases} .$$

we claim that every $h_t = h(t, \cdot)$ preserves $\mathbf{R}_-$, $[1, \infty[$, and $\mathfrak{h}^+$ and $\mathfrak{h}^-$, fixing each point of $[0, 1]$. We can now (since f realizes a homeomorphism of each triangle to $\mathfrak{h}^+$ or $\mathfrak{h}^-$ in a coherent way) define each H_t ($t \in [0, 1]$) as being the retraction h_t in each triangle (seen as the upper half plane if it is gray, the lower if it is white) without its vertex $\times$ (define it individually for each triangle and remark that they glue together). Then H is a homotopy relatively to E between a retraction $r : S \setminus p^{-1}(\{\infty\}) \to E = p^{-1}([0, 1])$ and identity, that is a retraction by deformation from $S \setminus p^{-1}(\infty)$ to the dessin E. □

Theorem 5.11. *Let (S, p) be a Belyi pair. Then the associated dessin is a tree if and only if $S \simeq \Sigma$ and p is polynomial.*

Proof. If the associated dessin is a tree, then by the above lemma $S' = S \setminus p^{-1}(\{\infty\})$ is simply connected so will be isomorphic to $\mathbb{D}$, $\mathbf{C}$, or Σ. Since $p_{|S'} : S' \to \mathbf{C}$ is surjective, continuous, and $\mathbf{C}$ is not compact, $S' \not\simeq \Sigma$. Take a holomorphic function f on S' that is bounded. Then we can build the elementary symmetric functions of f through p as in lemma 2.10. They are holomorphic functions on $\mathbf{C}$ that are bounded, so they must be constant, and f is constant too. Since there exist non constant holomorphic functions that are bounded in the disc, $S \simeq \mathbf{C}$. Since there is just one cell (otherwise there will exist a cycle in the dessin and it will not be a tree), there is only one point $x \in S$ above ∞. So $S = S' \cup \{x\} \simeq \mathbf{C} \cup \{\infty\} \simeq \Sigma$. Then p is a rational function with only one pole which is ∞, so it is a polynomial. □

Examples of dessins :

- The polynomial $p(z) = z^n$ gives us a dessin named the *star*.

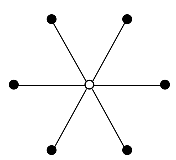

Figure 4: The star with 6 branches.

- The polynomial $f(z) = Cz^p(1 - z)^q$ with p and q two positive integers and C a normalisation constant gives us the *double star*. When computing the derivative, we see that f ramifies at the point $\frac{p}{p+q}$ of order 2 (and it is a $\bullet$ point) and at 0 of order p, at 1 of order q which are the two roots (so $\circ$ points) and then all other 1-values are no-ramification points.

- The rational function

$$g(z) = \frac{z^d + z^{-d} - 2}{4}\,,$$

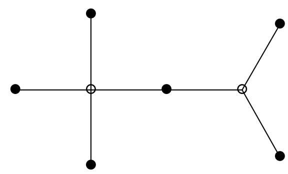

Figure 5: The double star with 4 and 3 branches

gives us a dessin which is not a tree, the $2d$-*gone*. The derivative $\frac{d}{4}(z^{2d}-1)z^{-d-1}$ vanish only at the $2d^{\text{th}}$ roots of unity, and in these points we will find $\circ$ points, the roots of g, which are the d^{th} roots of unity, and the others are $\bullet$ points.

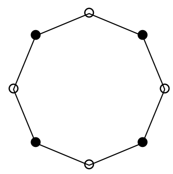

Figure 6: The octagon.

All these dessins have something in common : They are defined over $\mathbf{Q}$, and so the action of G on them is trivial. Let us find a non trivial example. Take the elliptic curve

$$Y^2 = X(X-1)(X-\sqrt{2})$$

and let us find an associated dessin. Like in the proof of the first part of Belyi's theorem we will apply the algorithm.
First start from $f = \mathbf{z}$ the projection on the first coordinate. It will ramify at $\{0, 1, \sqrt{2}, \infty\}$. Then compose with the minimal polynomial of $\sqrt{2}$ that is z^2-2. Its derivative is $2z$ so it ramifies only at $\{0\}$ (and $0^2-2=-2$). The ramification values are now $\{-2\} \cup \{-2,-1,0,\infty\}$ (the ramification values of the polynomial and the image by it of the preceding ramification values).
Then we want to eliminate ramification values to go to only $\{0, 1, \infty\}$. Take the affine function $z \mapsto \frac{z+2}{2}$ that will send -2 to 0, -1 to 1/2 and 0 to 1, and apply the Belyi polynomial with $p=1$ and $q=1$

$$R(z) = 4(\frac{z+2}{2})(1-\frac{z+2}{2}) = -z^2 - 2z$$

Its derivative is $-2z-2$ vanish only at -1 (and $R(-1)=1$) so our ramification values will be $\{1\} \cup \{0, 1, 0, \infty\} = \{0, 1, \infty\}$ and we are done.
Let us calculate the preimages of 0, 1 and ∞ step by step.

$$
\begin{array}{c|c|c}
0 & 1 & \infty \\
\uparrow & \uparrow & \uparrow \\
0, -2 & -1 & \infty \\
\uparrow & \uparrow & \uparrow \\
-\sqrt{2},\ \sqrt{2}, 0 & -1, 1 & \infty \\
\uparrow & \uparrow & \uparrow \\
\left(-\sqrt{2}, 2i\sqrt{\sqrt{2}-1}\right) & \left(-1, i\sqrt{2(1+\sqrt{2}}\right) & \infty \\
\left(-\sqrt{2}, -2i\sqrt{\sqrt{2}-1}\right) & \left(-1, i\sqrt{2(1+\sqrt{2}}\right) & \\
\left(\sqrt{2}, 0\right) & (0,0) &
\end{array}
$$

We can now draw the dessin on our surface step by step, making a particular construction from Σ to the elliptic curve. We just take two copies of Σ, then cut two holes on each (well chosen, here between 0 and 1 and between $\sqrt{2}$ and ∞) and glue the two copies along these holes. Do that construction is very useful to see the preimage of the dessin by the projection $(z, \zeta) \mapsto z$.

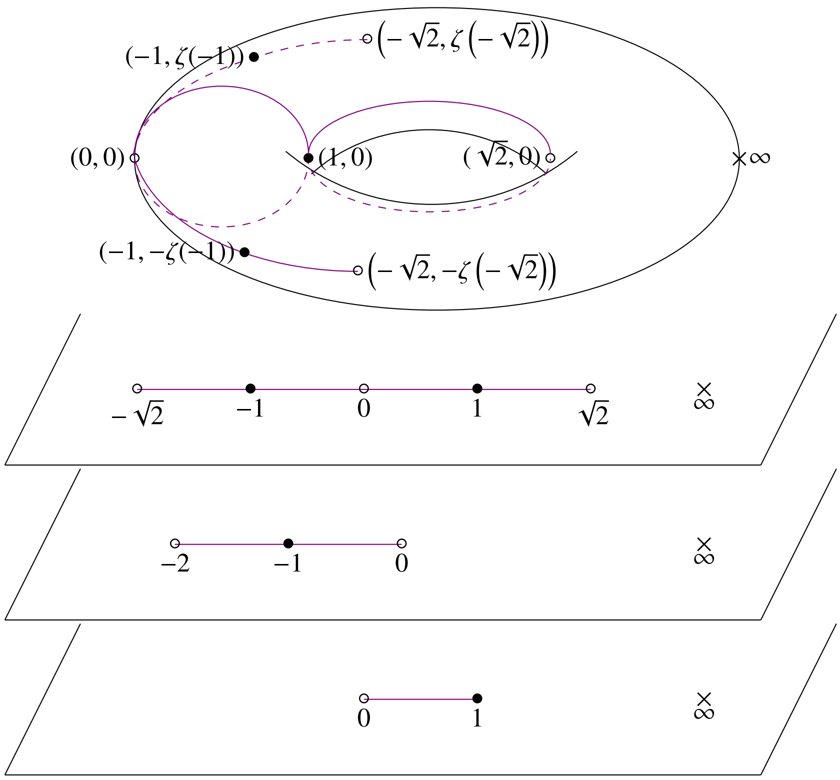

Figure 7: The dessin associated to $(z, \zeta) \mapsto -(z^2-2)^2 + 2(z^2-2)$ in the elliptic curve $Y^2 = X(X-1)(X-\sqrt{2})$ built step by step.

Then, applying the σ of G that permutes $\sqrt{2}$ and $(-\sqrt{2})$ we obtain the following dessin :

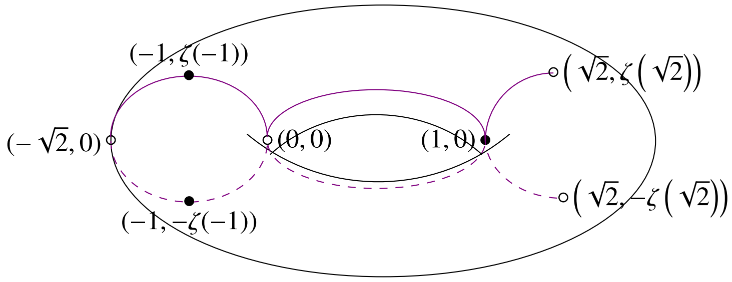

Figure 8: The dessin associated to $(z,\zeta) \mapsto -(z^2-2)^2 + 2(z^2-2)$ in the elliptic curve $Y^2 = X(X-1)(X+\sqrt{2})$.

The last has been built by cutting between $\sqrt{2}$ and 0 and between 1 and ∞. The two dessins are not isomorphic.

Invariants

Proposition 5.12. *Given a (equivalence class of) Belyi pair* (S, p) *and the dessin associated, the following properties are invariant by the action of* G*.*

- *The valency,*
- *The genus of the surface,*
- *The number of edges,*
- *The number of faces,*
- *The number of* $\circ$*-vertices and* $\bullet$ *vertices,*
- *The monodromy group of the cover,*
- *The cartographic group (in the sense of [9]) that is : the monodromy group of the dessin associated to* $4p(1-p)$*.*

Proof. We have already seen the invariance of the valency, it comes from the fact that be a order n ramification value means vanish some polynomials and not vanish others and these properties are invariants by G. The genus, and number of edges, faces and vertices are the same because they are encoded in the valency. For the monodromy and the cartographic groups it is quite long and one can look at [9]. □

References

In addition to all the references made in the article, we can cite :
- [15] with another point of view on Belyi's theorem,
- [10] with some applications of the theory,
- [14], [13] and [12] from which some articles that I used are taken, that are made with talks about the dessins, the inverse Galois problem, moduli spaces, etc.,
- [2] the paper of Belyi where he proved the theorem, with an application to the inverse Galois problem.

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